

Wightman Functions' Behaviour on the Event Horizon of an Extremal Reissner-Nordström Black Hole.

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May - 1995

Abstract:

A weaker Haag, Narnhofer and Stein prescription as well as a weaker Hessling Quantum Equivalence Principle for the behaviour of thermal Wightman functions on an event horizon are analysed in the case of an extremal Reissner-Nordström black hole in the limit of a large mass. In order to avoid the degeneracy of the metric in the stationary coordinates on the horizon, a method is introduced which employs the invariant length of geodesics which pass the horizon.

First the method is checked for a massless scalar field on the event horizon of the Rindler wedge, extending the original procedure of Haag, Narnhofer and Stein onto the whole horizon and recovering the same results found by Hessling.

Afterwards the HNS prescription and Hessling's prescription for a massless scalar field are analysed on the whole horizon of an extremal Reissner-Nordström black hole in the limit of a large mass. It is proved that the weak form of the HNS prescription is satisfied for all the finite values of the temperature of the KMS states, i.e., this principle does not determine any Hawking temperature. It is found that the Reissner-Nordström vacuum, i.e., $T = 0$ does satisfy the weak HNS prescription and it is the only state which satisfies weak Hessling's prescription, too. Finally it is suggested that all the previously obtained results should be valid dropping the requirements of a massless field and of a large mass black hole, too.

PACS number(s): 04.62.+v, 04.70.Dy, 11.10.Cd, 11.10.Wx

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Introduction

Recently Hawking, Horowitz and Ross [1] have discussed the thermodynamics of an extremal Reissner-Nordström black hole. It seems to follow from this discussion that thermal states in thermal equilibrium outside of it may have every value of temperature because the extremal R-N black hole may have every value of temperature, too.

In literature there exist different methods of seeking the possible temperatures of thermal states of free scalar fields in a static globally hyperbolic space-time region with horizons. The most common and popular method consists of two mathematical steps (see for example [2]).

At the beginning one has to extend the time coordinate to imaginary values and eliminate all the metric singularities connected to the horizon by an opportune choice of the imaginary time periodicity β_M . The second step is to impose the KMS condition for thermal states [3, 4], i.e., to impose the periodicity condition on the imaginary time dependence of the thermal Wightman functions and interpret the common period β_T as $1/T$, where T is the temperature of the state. Note that, because of the time periodicity of the manifold, it is not possible to fix the value of β_T arbitrarily, but the permitted values must be of the form $\beta_n = \beta_M/n$, where $n = 1, 2, 3, \dots$. The value obtained for $n = 1$, i.e., the periodicity of the imaginary time manifold $\beta_1 = \beta_M$ determines the Hawking temperature: $T_H = 1/\beta_1 = 1/\beta_M$.

However it is important to stress that the integers $n = 2, 3, 4, \dots$ produce also correct periodic Wightman functions on the same Euclidean manifold (see [5] on similar topics) and *a priori* there is no reason to reject these additional thermal states in absence of any other physical requirements as, for example, some regularity prescription for the renormalized stress-tensor.

A second method, which from now on will be called the HNS principle, was introduced by Haag, Narnhofer and Stein in 1984 [6] (see also [7, 4]) and successively developed by Hessling [8]. This method is connected to the well known Hadamard expansion of two-point Green functions in a curved space-time in the limit of the coincidence of the arguments. Haag, Narnhofer and Stein in [6] proved that if one assumes fairly standard axioms of quantum (quasi-free) field theory, particularly *local definiteness* and *local stability* in an at least stationary, causally complete space-time region, then, roughly speaking, the thermal Wightman functions in the *interior* of this region will transform into *non thermal* and *massless* Wightman functions in the flat space-time when the “distance” of the arguments is vanishing (see the formula below).

We will call this statement, which from a naive point of view seems to follow from the Einstein equivalence principle, the HNS *theorem*.

The statement above is valid also in the case $T = 0$. Furthermore, one should note that within the framework of the HNS theorem the Wightman functions are properly considered to be *distributions*. For example, in case of two-point thermal Wightman functions of a scalar field it holds:

$$\lambda^2 W_\beta^\pm(x + \lambda z_1, x + \lambda z_2) \rightarrow \frac{1}{4\pi^2} \frac{1}{g_{\mu\nu}(x) z^\mu z^\nu} \quad \text{as } \lambda \rightarrow 0^+, \quad (1)$$

where $z = z_2 - z_1$. In the equation above the coordinates $x \equiv x^\mu$ indicate a point in the *interior* of the region,

$$z_{(j)} \equiv z_{(j)}^\mu \frac{\partial}{\partial x^\mu} \Big|_x$$

indicates vectors in the tangent space at x and finally we used the obvious notation:

$$x + z_{(j)} \equiv x^\mu + z_{(j)}^\mu .$$

Both sides of Eq.(1) are distributions acting on a couple of smooth test functions in the corresponding variables z_1 and z_2 .

Finally Haag, Narnhofer and Stein *principle* generalises HNS theorem and it affirms that in the case the space-time region we are dealing with is just *a part* of the whole manifold separated by *event horizons*, the point coincidence behaviour of the Wightman functions for a physically sensible (thermal or not) state must hold also *onto the horizons*.

Haag Narnhofer and Stein proved in [6] that in the case of Rindler and Schwarzschild space-times, this constraint holds only for $\beta_T = \beta_M$, the same value obtained by the first method. The HNS principle determines the Unruh and Hawking temperatures.

Actually, the above statement holds even if a weaker version of HNS principle is used. Indeed one must be careful in using literally Eq.(1) because therein the time component of the vector z is endowed with a small imaginary part $\mp i\varepsilon$ and it is understood the ε -prescription which involves two weak limits as $\varepsilon \rightarrow 0^+$ first and $\lambda \rightarrow 0^+$ afterward. So Haag Narnhofer and Stein, on their way to find the Hawking temperature, interpreted the Wightman functions in Eq. (1) strictly as functions. In this sense Eq. (1) implies:

$$\lambda^2 W_\beta^\pm(x, x + \lambda z) \rightarrow \frac{1}{4\pi^2} \frac{1}{g_{\mu\nu}(x) z^\mu z^\nu} \quad \text{as } \lambda \rightarrow 0^+ , \quad (2)$$

In fact it can be simply proved that the stronger version of the HNS principle implies the weaker one, formally expressed by Eq. (2), by using smooth test functions $f(z_1)$ and $f(z_2)$ in (1) which are not “light-like correlated”, i.e., such that all the vectors $z = z_2 - z_1$ are not *light-like* whenever $f(z_1) \neq 0$ and $f(z_2) \neq 0$. Generally speaking, this interpretation does not eliminate the ε -prescription, as it is also necessary to deal with possible cuts in the complex time plane, but it transforms the weak limits in Eq. (1) into usual limits of functions.

Following the original paper of Haag Narnhofer and Stein we interpret Eq. (1) in this weaker sense and thus we evaluate the limit in Eq. (2) for *real* vectors z , space-like or time-like; for light-like vectors we expect a divergent limit².

In order to use the (weak) HNS principle for two-point Wightman functions one has to check their behaviour as one point is fixed on the horizon and the other is running toward the first from the inside of the considered region. This is not as simple as one might think at first, because the metric could become degenerated on the horizon in the stationary coordinates which define the studied thermal Wightman functions, consequently it could not be possible to write down the right hand side of Eq. (2) in that coordinate frame. However, as pointed by Haag, Narnhofer and Stein in [6], one can check the validity of the HNS principle in stationary coordinates using directly Eq. (2) for specific points on the horizons, i.e., for those which belong to the intersection of the past and the future horizons, and along appropriate directions, but not on the whole horizon. It is very interesting to note that the check of the behaviour of Wightman functions in these “few” points is sufficient to determine the Unruh and the Hawking temperatures respectively in the case of the Rindler wedge and the Schwarzschild background. Hessling proved

²In our massless case this is equivalent to work with the Hadamard function instead of the Wightman functions, see below.

in [8] that the HNS principle, in the case of the Rindler wedge, determines the Unruh temperature only when it works on the intersection of the horizons, but it does not determine any temperature by considering the remaining points. In **Section 1** we will report an independent proof of this fact.

Unfortunately in the case of an extremal Reissner-Nordström black hole the past and the future horizons do not intersect and furthermore it is not possible to deal with the original procedure of Haag, Narnhofer and Stein because in this case some important technical hypothesis does not hold, e.g., the requirement of a non-vanishing surface gravity or related parameters [6], consequently, up to now, there is no proof of the validity of the HNS principle for some (or every!) value of β in this case.

In this situation the Hessling development of the HNS principle results to be very useful because, in the Minkowski space-time at least, it determines the Unruh temperature working also considering the points which do not belong to the intersection of the horizons [8]. In **Section 1** we will report an independent proof of this fact, too. We may expect a similar result in the case of an extremal R-N black hole where the intersection does not exist.

Furthermore [8] one could note that, in a real (Schwarzschild) black hole, the *past* event horizon does not exist and thus also the intersection of horizons does not exist. For this reason the Hessling principle result to be very important as far as the possibility to use this in more physical situations than the eternal black hole cases is concerned.

Hessling's principle, trying to define a *Quantum (Einstein) Equivalence Principle* to be imposed on the physically sensible quantum states [8]³, requires the *existence* of the limit:

$$\lim_{\lambda \rightarrow 0^+} N(\lambda)^2 W_{\beta}^{\pm}(x + \lambda z_1, x + \lambda z_2), \quad (3)$$

as a *continuous* function of x , for some function $N(\lambda)$ monotonous and nonnegative for $\lambda > 0$. Furthermore, it requires the validity of a much more strong condition in every *local inertial coordinate system* around x (i.e., a coordinate frame such that $g_{\mu\nu}(x) = \text{diag}(-1, 1, 1, 1)$, $\partial_{\rho} g_{\mu\nu}(x) = 0$):

$$\lim_{\lambda \rightarrow 0^+} \frac{d}{d\lambda} N(\lambda)^2 W_{\beta}^{\pm}(x + \lambda z_1, x + \lambda z_2) = 0. \quad (4)$$

The latter requirement is connected with the fact that within a local inertial system the metric looks like the Minkowski metric *up to the first order* in the coordinate derivatives. In the Minkowski background and using Minkowskian coordinates Eq.(4), which involves coordinate derivatives *up to the first order*, results to be satisfied and thus we expect this will hold in curved backgrounds by using local inertial coordinate systems⁴, too (see [8] for details). We can note that, away from the horizons, the validity HNS *theorem* implies the validity of the first Hessling requirement with $N(\lambda) = \lambda$. Furthermore, the validity of the HNS principle on a horizon implies the validity of the first Hessling requirement there.

As in the case of the HNS principle, we will use the Hessling principle on an event horizon in a *weaker* version, by checking the validity of Eq.(2) as well as of the following equation:

$$\frac{d}{d\lambda} \lambda^2 W_{\beta}^{\pm}(x, x + \lambda z) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow 0^+, \quad (5)$$

³Really, Hessling considers in [8] also n -point functions, but we will restrict our discussion by considering only the case of a scalar quasifree field and thus by studying only the two-point functions.

⁴We stress that Hessling generalises the above equations in order to be able to use a non local inertial coordinate system [8], too.

where x belongs to the horizon, z is a *real space-like or time-like* vector and a *local inertial coordinate system* is used.

It is interesting to check whether the HNS principle and the Hessling principle select special temperatures or, like the method mentioned first, accept every temperature for thermal states in the case of an extremal Reissner-Nordström black hole. This question arises also because in the case of the Rindler and Schwarzschild space-times the use of the HNS and Hessling principle seems to be more selective than the usually used first method. Indeed, employing the HNS (or the Hessling) principle, one obtains the result that all the temperatures of the form $T_n = n/\beta_M$ with $n = 2, 3, \dots$ must be rejected which otherwise would be permitted⁵.

Finally, it seems necessary to spend some words about the very important *Kay-Wald theorem* [9]. They proved that in a space-time with a *bifurcate Killing horizon* [9] (furthermore endowed with an opportune discrete “wedge reflection” isometry) like the Minkowski manifold endowed with Rindler’s wedges as well as the Kruskal space-time endowed with Schwarzschild wedges, every quasifree *stationary* state (with respect to the Killing vector defining the horizons) satisfying the *Hadamard condition*[9, 4] in a neighborhood of the horizon, results to be a KMS state inside of the wedges where the Killing vector is time-like. Furthermore the temperature results to be the Unruh-Hawking temperature, too.

We want to stress that in the background of an extremal Reissner-Nordström black hole it is impossible to use the Kay-Wald theorem because it needs explicitly a *not empty* intersection of horizons [9].

In **Section 1** we will start with some aspects of the well known Rindler theory to observe some interesting features of the HNS principle and the Hessling principle by extending the “horizon-check” onto the whole horizon recovering the same results found by Hessling [8].

In **Section 2**, by using this generalised method, we will study the temperatures of thermal states in the case of a extremal Reissner-Nordström black hole for a scalar field and in the limit of a black hole with large mass. We will take advantage there of the vanishing scalar curvature of the manifold and we will use the resulting coincidence of the conformal coupling with the minimal coupling for a massless scalar field.

Finally, in **Section 3** we will discuss our results.

1 HNS and Hessling’s principles in the Rindler Space

We consider the Minkowski space-time with signature:

$$g_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1) .$$

In this manifold we consider a local coordinate frame (ρ, τ, x^\perp) connected to Minkowski coordinates $(x^0, x^1, x^2, x^3) = (t, \mathbf{x})$ by the equations:

$$x^0 = \rho \sinh \tau , \tag{6}$$

$$x^1 = \rho \cosh \tau , \tag{7}$$

⁵Actually, as for example in the case of the Rindler space, other physical requirements (e.g., the behaviour of the stress-tensor on horizon) reject these temperature too.

$$x^\perp = (x^2, x^3), \quad (8)$$

where $\tau, x^2, x^3 \in (-\infty, +\infty)$ and $\rho > 0$.

This region is called *Rindler wedge* [10]. In Minkowski coordinates it is:

$$x^1 > |x^0|.$$

In Rindler coordinates, the Minkowski metric is written as:

$$ds^2 = -\rho^2 d\tau^2 + d\rho^2 + (dx^\perp)^2. \quad (9)$$

Note that the part:

$$H^+ = \{x^1 = x^0, x^1 \geq 0\} \quad (10)$$

of the boundary of the Rindler wedge is the *future* event horizon and the part:

$$H^- = \{x^1 = -x^0, x^1 \leq 0\} \quad (11)$$

is the *past* event horizon of the region.

Now we calculate the Wightman functions (see [11] for example) and the thermal Wightman functions (see [10] for example) of a massless field and then discuss the HNS principle in this simple case.

To construct the canonical theory [11] of a massless scalar field propagating in the Rindler wedge we expand the field operator into positive and negative frequency modes of the time-like Killing vector field tangent to the τ coordinate. This vector field also generates τ -translations.

By this way we obtain:

$$\phi = \int \frac{d^2 k}{2\pi} \int_0^\infty \frac{d\omega}{\pi} [\sinh(\pi\omega)]^{1/2} K_{i\omega}(k\rho) \left[A_{k\omega} e^{i(kx^\perp - \omega\tau)} + A_{k\omega}^\dagger e^{-i(kx^\perp - \omega\tau)} \right]. \quad (12)$$

Then we impose the commutation relations which are equivalent to the usual canonical commutation rules:

$$\begin{aligned} [A_{k\omega}, A_{k'\omega'}] &= 0, \\ [A_{k\omega}, A_{k'\omega'}^\dagger] &= \delta(k - k') \delta(\omega - \omega'), \\ [A_{k\omega}^\dagger, A_{k'\omega'}^\dagger] &= 0. \end{aligned} \quad (13)$$

It is easy to see that this fact follows from the orthogonality and completeness relations:

$$\int_0^\infty K_{i\omega}(k\rho) K_{i\omega'}(k\rho) \rho^{-1} d\rho = \frac{\pi^2}{2\omega \sinh \pi\omega} \delta(\omega - \omega'), \quad (14)$$

$$\int_0^\infty \frac{2\omega \sinh \pi\omega}{\pi^2} K_{i\omega}(k\rho) K_{i\omega'}(k\rho') d\omega = \rho \delta(\rho - \rho'). \quad (15)$$

$K_{i\omega}(k\rho)$ denotes the Mc Donald functions of imaginary order $i\omega$.

We obtain the Wightman elementary functions by calculating the bra-ket average of a two field operator product in the vacuum state, annihilated by $A_{k\omega}$ [11]. They are:

$$W^+(x, x') = \int \frac{d^2k}{4\pi^2} e^{ik(x^\perp - x'^\perp)} \int_0^\infty \frac{d\omega}{\pi^2} \sinh \pi\omega e^{-i\omega(\tau - \tau' - i\varepsilon)} K_{i\omega}(k\rho) K_{i\omega}(k\rho'), \quad (16)$$

$$W^-(x, x') = \int \frac{d^2k}{4\pi^2} e^{ik(x^\perp - x'^\perp)} \int_0^\infty \frac{d\omega}{\pi^2} \sinh \pi\omega e^{i\omega(\tau - \tau' + i\varepsilon)} K_{i\omega}(k\rho) K_{i\omega}(k\rho'), \quad (17)$$

where we used the abbreviation $x = (\tau, \rho, x^\perp)$.

It is well known that the other Green functions can be built up with the Wightman elementary functions [11].

We can integrate the expression for $W^\pm(x, x')$ and obtain [14, 15]:

$$W^\pm(x, x') = -\frac{1}{4\pi^2} \frac{\alpha}{\rho\rho' \sinh \alpha} \frac{1}{(\tau - \tau' \mp i\varepsilon)^2 - \alpha^2}, \quad (18)$$

where:

$$y = \frac{\rho^2 + \rho'^2 + |x^\perp - x'^\perp|^2}{2\rho\rho'} \quad \text{and} \quad \cosh \alpha = y. \quad (19)$$

Note that W^\pm are distributions.

It is important to stress that, for $\varepsilon = 0$ and $\rho, \rho', x^\perp, x'^\perp$ fixed, the right hand side of Eq. (18), strictly considered as a function, can be analytically extended to a function W defined in the whole complex $(\tau - \tau')$ -plane except at the poles $(\tau - \tau') = \pm\alpha$. It is also important to note that we would have obtained exactly the same extension by starting from W^- or W^+ because of the triviality of equal-time commutation relations. In terms of W Eq. (18) reads:

$$W^\pm(\tau - \tau') = W(\tau - \tau' \mp i\varepsilon),$$

where $\tau - \tau'$ is real valued now. The ε -prescription indicates the manner in which one obtains distributions W^+ and W^- as weak limits. According to the KMS condition for bosons [3], this extended thermal Wightman function W_β must be periodic in imaginary time with period β . One can find W_β with the *method of images* [12, 10]. It reads:

$$W_\beta(x, x') = -\frac{1}{4\pi^2} \frac{\alpha}{\rho\rho' \sinh \alpha} \sum_{n=-\infty}^{\infty} \frac{1}{(\tau - \tau' - in\beta)^2 - \alpha^2}.$$

We can sum the series, the result being:

$$W_\beta(x, x') = \frac{1}{4\pi^2} \frac{\pi \left\{ \coth \frac{\pi}{\beta}(\alpha + \tau - \tau') + \coth \frac{\pi}{\beta}(\alpha - \tau + \tau') \right\}}{2\beta\rho\rho' \sinh \alpha}.$$

To make it clear, this expression indicates the whole complex extension of *both* of the thermal Wightman functions. In our massless case this function with $\tau - \tau'$ real valued coincides with the *Hadamard function* $W^{(1)} := W^+ + W^-$ [11, 10] except for a factor 1/2.

Finally, one return to the real time thermal Wightman functions in the *distributional* sense by restoring the usual ε -prescription obtaining [15]:

$$W_{\beta}^{\pm}(x, x') = \frac{1}{4\pi^2} \frac{\pi \left\{ \coth \frac{\pi}{\beta}(\alpha + \tau - \tau' \mp i\varepsilon) + \coth \frac{\pi}{\beta}(\alpha - \tau + \tau' \pm i\varepsilon) \right\}}{2\beta\rho\rho' \sinh \alpha}. \quad (20)$$

In this *massless* case the singularities at $\tau - \tau' = \pm\alpha$ are poles and therefore it is not necessary to use the ε -prescription for the functions, but if we deal with massive fields these poles become branching points [12]. In this situation the ε -prescription, in case of causally related arguments x and x' , tells us how to calculate the limit, depending on the side from which we approach the cuts, in order to distinguish W_{β}^{-} from W_{β}^{+} .

In order to find the Unruh temperature we should note that the metric could become singular if we extend τ to imaginary values: $\tau \rightarrow -i\tau$

$$ds^2 = \rho^2 d\tau^2 + d\rho^2 + (dx^{\perp})^2. \quad (21)$$

Indeed, let β be the period of the imaginary time coordinate. If $\beta \neq 2\pi$ then the metric will have a non trivial *conical*-like singularity at $\rho = 0$ (however it is possible to study the quantum field theory also in this background [16, 18, 19, 20]). Thus we see that $\beta = 2\pi$ is the only choice in order to have a globally regular manifold⁶. This fact implies that the period of the thermal green functions must be of the kind:

$$\beta_k = \frac{2\pi}{k}, \quad (22)$$

where $k = 1, 2, 3, \dots$. It means that *a priori* the possible temperatures are just of the form:

$$T_k = \frac{k}{2\pi}, \quad (23)$$

where $k = 1, 2, 3, \dots$

The value: $T_U = T_1 = \frac{1}{2\pi}$ is the well known *Unruh temperature* (for complete references see: [21, 4, 10]).

After some algebra with the hyperbolic functions, we recover the well known result for $\beta_U = \beta_1 = 2\pi$, holding inside of a Rindler wedge and in the sense of the distributions:

$$\begin{aligned} W_{2\pi}^{\pm}(x, x') &= \frac{1}{4\pi^2} \frac{1}{\rho^2 + \rho'^2 + |x^{\perp} - x'^{\perp}|^2 - 2\rho\rho' \cosh(\tau - \tau' \mp i\varepsilon)} \\ &= \frac{1}{4\pi^2} \frac{1}{-(t - t' \mp i\varepsilon)^2 + (x^1 - x'^1)^2 + |x^{\perp} - x'^{\perp}|^2} \\ &= \frac{1}{4\pi^2} \frac{1}{\sigma^2(x_{\mp\varepsilon}, x')}, \end{aligned} \quad (24)$$

where we used the notation $x_{\mp\varepsilon} \equiv (t \mp i\varepsilon, \mathbf{x})$ and introduced the geodesic invariant distance:

$$\sigma^2(x, y) = -(t_x - t_y)^2 + (x_x^1 - x_y^1)^2 + |x_x^{\perp} - x_y^{\perp}|^2 \quad (25)$$

$$= -(t_x - t_y)^2 + |\mathbf{x}_x - \mathbf{x}_y|^2. \quad (26)$$

⁶This requirement arises when one tries to use the functional integral over the gravitational configurations as well as over the quantum field configurations [17].

$W_{2\pi}^+$ is exactly the Wightman function of the Minkowski vacuum (thus it satisfies the HNS and Hessling prescriptions everywhere trivially). This means that the Minkowski vacuum is a KMS state with respect to τ -translations. There exists a vast literature about this topic; for a complete review see [4] and [21].

Now we will check the HNS and the Hessling principle on the whole horizon for arbitrary values of β . Incidentally, it shall be mentioned that one can check the correct behaviour of Wightman functions away from the horizons with no particular difficulties but in this paper we will deal with the Wightman functions behaviour on the horizons exclusively.

Note that in Rindler coordinates the coordinate representation of the metric (but not the metric) becomes degenerated onto the horizons, so there seems to be the necessity to introduce new, generally non stationary, coordinate frames in a neighborhood of every point of the horizons, to check the HNS and Hessling principles. The price that must be payed when one uses non stationary coordinates is that one has to drop the translational time-invariance of the Wightman (and Green) functions and thus the theory becomes more complicated. (Actually, as pointed out in the introduction, the use of new coordinates is not necessary for the points which belong to $H^+ \cap H^-$ [6], but in our aim to be more general, we want to deal with the whole horizon).

In the Rindler space-time one could use Minkowski coordinates, but in other curved space-times it is rather unlikely to find similar simple coordinate frames! Thus we will develop a method which employs only stationary coordinates wherein the thermal theory is much more simple.

Indeed it is possible to check the HNS and the Hessling principles by studying the behaviour of two point Wightman functions along every geodesic which starts from the event horizons. We note that for every geodesic which meets the horizon in a point x there is a locally geodesic coordinate frame with the origin at x . Furthermore we can chose the geodesic to be a coordinate axis, the running coordinate being just the length of the geodesic measured from x . By varying the geodesics which meet the horizon one obtains all the possible points of the horizon together with their tangent vectors. We stress that it is possible to execute this checking procedure also by using stationary coordinates (Rindler coordinates in this case), because the geodesic length is invariant and so is not sensitive to the degeneracy in this representation of the metric.

Let us illustrate this method for the Rindler case.

For sake of simplicity we will only deal with geodesics in a plane $x^\perp = \text{constant}$. For geodesics which meet the horizon we obtain some useful formulas by translating the linear geodesic equations from Minkowski coordinates into Rindler coordinates.

Geodesics which meet $H^+ \cap H^-$.

$$\tau = \tanh^{-1} \alpha \tag{27}$$

$$\rho = s \tag{28}$$

$$s = x^1 \sqrt{1 - \alpha^2}, \tag{29}$$

$$\tag{30}$$

where $\alpha \in (-1, +1)$ is a constant parameter and s is the geodesic length measured from

the horizon. It can be easily proved that all these geodesics are space-like.

Space-like geodesics which meet $H^+ - (H^+ \cap H^-)$.

$$\tau = \gamma + \coth^{-1} \left(1 + \frac{se^\gamma}{y} \right) \left(= \gamma + \sinh^{-1} \left(\frac{ye^{-\gamma}}{\rho} \right) \right) \quad (31)$$

$$\rho = ye^{-\gamma} \sqrt{\left(1 + \frac{se^\gamma}{y} \right)^2 - 1} \left(= \frac{ye^{-\gamma}}{\sinh(\tau - \gamma)} \right) \quad (32)$$

$$s = ye^{-\gamma} (\coth(\tau - \gamma) - 1) , \quad (33)$$

where $\gamma \in (-\infty, +\infty)$ is a constant parameter, s is the geodesic length measured from the horizon and $y = x^1 = x^0$ is the coordinate of the intersection of the geodesic and H^+ .

Time-like geodesics which meet $H^+ - (H^+ \cap H^-)$.

$$\tau = \gamma + \tanh^{-1} \left(1 - \frac{se^\gamma}{y} \right) \left(= \gamma + \cosh^{-1} \left(\frac{ye^{-\gamma}}{\rho} \right) \right) \quad (34)$$

$$\rho = ye^{-\gamma} \sqrt{1 - \left(1 - \frac{se^\gamma}{y} \right)^2} \left(= \frac{ye^{-\gamma}}{\cosh(\tau - \gamma)} \right) \quad (35)$$

$$s = ye^{-\gamma} (1 - \tanh(\tau - \gamma)) , \quad (36)$$

where $\gamma \in (-\infty, +\infty)$ is a constant parameter, s is the geodesic length measured from the horizon and $y = x^1 = x^0$ is the coordinate of the intersection between the geodesic and H^+ .

One can find similar formulas for geodesics which meet H^- , but we deal only with geodesics falling into H^+ because of the trivial time-symmetry of the problem. In fact the Minkowski time-reversal transformation:

$$t \rightarrow -t$$

$$\mathbf{x} \rightarrow \mathbf{x}$$

is equivalent to the Rindler time-reversal transformation:

$$\tau \rightarrow -\tau$$

$$\rho \rightarrow \rho$$

$$x^\perp \rightarrow x^\perp$$

One obtains the geodesics meeting H^- by using the above time transformation into geodesics which meet H^+ . Furthermore the same time-reversal action transforms the Wightman functions of Eq. (20) into their complex conjugate. In this way can be easy proved that the Wightman functions will satisfy HNS and Hessling's prescription on H^- if they do so on H^+ .

We start studying the geodesics which run through the origin of Minkowski coordinates. We obtain, by inserting (27) and (28) into (19) and (20) and by calculating the limit as $s = \rho \rightarrow 0$:

$$W_{\beta}^{\pm}(x_{H^+}, x') = \frac{1}{4\pi^2} \frac{2\pi}{\beta \rho'^2},$$

where $x_{H^+} \equiv (x^0 = 0, x^1 = 0, x^2, x^3)$.

Using the same notations as in Eq. (2), we substitute $\rho' = s$ thinking $s \equiv z^i$ to be an increase of a space-like coordinate of a coordinate frame in x_{H^+} , as well as to be the only non-vanishing component of a tangent vector z in x_{H^+} . The result reads:

$$\begin{aligned} \lambda^2 W_{\beta}^{\pm}(x_{H^+}, x_{H^+} + \lambda z) &= \lambda^2 W_{\beta}(x_{H^+}, x_{H^+} + \lambda z) = \\ &= \lambda^2 \frac{1}{4\pi^2} \frac{2\pi}{\beta \lambda^2 z^{i2}} = \frac{1}{4\pi^2} \frac{2\pi}{\beta} \frac{1}{g_{\mu\nu}(x_{H^+}) z^{\mu} z^{\nu}}. \end{aligned}$$

Thus we see that, in order to be consistent with the HNS principle, $\beta = 2\pi$ is the only possible choice. Before to consider the other cases we note that if we tried to repeat the calculations above with the *zero temperature* green function (18), we would fall in trouble. In fact, as $x \rightarrow H^+ \cap H^-$ along the considered geodesics:

$$W^{\pm}(x, x') \rightarrow 0,$$

for every x' . We conclude that the requirement of the validity of the HNS principle rejects also the *Rindler vacuum* which for that can not be considered as a physically sensible state for the whole Minkowski space-time⁷. Note that the Rindler vacuum is outside of the Fock representation of Hilbert space generated by Minkowski vacuum and *vice versa*. Indeed it is well known that the Minkowski vacuum requires a vanishing normalization coefficient when it is built up by the normal modes of the Fock representation generated by the (left hand and right hand) Rindler vacuum [21].

However within the framework of the usual *algebraic approach* to quantum field theory [4, 9] such a situation is quite common and, differently from the HNS prescription, it does not distinguish directly between physical and unphysical states.

Now we consider the case of the space-like geodesics which fall into H^+ , in a point $x_{H^+} \equiv (y, y, y^{\perp})$.

We obtain by inserting (31) and (32) into (19) and (20) and by calculating the limit as $s = \rho \rightarrow 0$:

$$W_{\beta}^{\pm}(x_{H^+}, x') = \frac{1}{4\pi^2} \frac{\pi}{\beta \rho'^2} \left\{ 1 + \coth \left[\frac{\pi}{\beta} \left(\tau' + \ln \frac{\rho'}{2y} \right) \right] \right\}. \quad (37)$$

It is important to stress that this function diverges for:

$$\tau' + \ln \frac{\rho'}{2y} = 0,$$

⁷ If we consider rather the Rindler wedge as the whole physical manifold, then we can consider the Rindler vacuum as a physical state. Note that in such a situation the prescriptions for renormalizing physical quantities also change, because it is not opportune to subtract the corresponding Minkowski quantities.

i.e., on *null* geodesics which fall in x_{H^+} , just as we have expected.

Let us examine the Wightman function behaviour on space-like geodesics.

Using again (31) and (32) for the variables τ' and ρ' , substituting $s = \lambda z^i$ and finally calculating the limit as $\lambda \rightarrow 0$ we obtain ($W^\pm = W$):

$$\begin{aligned}
\lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) &= \frac{1}{4\pi^2} \frac{\pi \lambda \left\{ 1 + \coth \left[\frac{\pi}{\beta} \ln \left(1 + \frac{\lambda z^i e^\gamma}{2y} \right) \right] \right\}}{\beta e^{-\gamma} y z^i \left(2 + \frac{\lambda z^i e^\gamma}{y} \right)} \\
&\sim \frac{1}{4\pi^2} \frac{\pi}{2\beta e^{-\gamma} y z^i} \frac{\lambda}{\sinh \left(\frac{\pi}{\beta} \ln \left(1 + \frac{\lambda z^i e^\gamma}{2y} \right) \right)} \\
&\sim \frac{1}{4\pi^2} \frac{\pi}{2\beta e^{-\gamma} y z^i} \frac{\lambda}{\frac{\pi}{\beta} \ln \left(1 + \frac{\lambda z^i e^\gamma}{2y} \right)} \\
&\sim \frac{1}{4\pi^2} \frac{1}{2e^{-\gamma} y z^i} \frac{\lambda}{\frac{\lambda z^i e^\gamma}{2y}} \\
&= \frac{1}{4\pi^2} \frac{1}{z^{i2}} = \frac{1}{4\pi^2} \frac{1}{g_{\mu\nu}(x_{H^+}) z^\mu z^\nu}.
\end{aligned} \tag{38}$$

We found that the HNS principle holds for this kind of geodesics for all values $\beta > 0$.

Finally we consider the case of time-like geodesics meeting H^+ .

After some limit calculations as $s \rightarrow 0$ in the first variable x of Wightman function (20) we obtain, inserting (34) and (35) into (19) and (20):

$$W_\beta^\pm(x_{H^+}, x') = \frac{1}{4\pi^2} \frac{\pi}{\beta \rho'^2} \left\{ 1 + \coth \left[\frac{\pi}{\beta} \left(\tau' + \ln \frac{\rho'}{2y} \right) \right] \right\}$$

This expression for W_β^\pm coincides with the right hand side of Eq. (37). Indeed the value of a Wightman function with one point on the horizon must not depend on the direction from which we reach the horizon! Now, by using again (34) and (35) for the variables τ' and ρ' , by substituting $s = \lambda z^0$ with $\lambda \rightarrow 0$ we obtain ($W^\pm = W$):

$$\begin{aligned}
\lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) &= \frac{1}{4\pi^2} \frac{\pi \lambda \left\{ 1 + \coth \left[\frac{\pi}{\beta} \ln \left(1 - \frac{\lambda z^0 e^\gamma}{2y} \right) \right] \right\}}{\beta e^{-\gamma} y z^0 \left(2 - \frac{\lambda z^0 e^\gamma}{y} \right)} \\
&\sim \frac{1}{4\pi^2} \frac{\pi}{2\beta e^{-\gamma} y z^0} \frac{\lambda}{\sinh \left(\frac{\pi}{\beta} \ln \left(1 - \frac{\lambda z^0 e^\gamma}{2y} \right) \right)} \\
&\sim \frac{1}{4\pi^2} \frac{\pi}{2\beta e^{-\gamma} y z^0} \frac{\lambda}{\frac{\pi}{\beta} \ln \left(1 - \frac{\lambda z^0 e^\gamma}{2y} \right)} \\
&\sim -\frac{1}{4\pi^2} \frac{1}{2e^{-\gamma} y z^0} \frac{\lambda}{\frac{\lambda z^0 e^\gamma}{2y}} \\
&= -\frac{1}{4\pi^2} \frac{1}{z^{02}} = \frac{1}{4\pi^2} \frac{1}{g_{\mu\nu}(x_{H^+}) z^\mu z^\nu}.
\end{aligned} \tag{39}$$

Once more the HNS principle holds for the considered points on the horizon and for all the values $\beta > 0$. We conclude that *only the points in $H^+ \cap H^-$ really select a temperature of thermal states* in the framework of the *HNS principle*.

Let us consider the *Hessling principle* which will produce a quite different result (but consistent with the previous one).

In the following we will define:

$$\mu := Z \frac{e^\gamma}{2y} \lambda ,$$

where we may understand either $Z = z^i$ or $Z = -z^0$. Furthermore, we define:

$$X := \frac{\pi}{\beta} \ln(1 + \mu) .$$

Considering Eq.(38) as well as Eq.(39) in order to check Hessling's principle for points on the horizon which do not belong to $H^+ \cap H^-$, we obtain:

$$\frac{d}{d\lambda} \lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) = \frac{A}{\beta e^{2\beta X/\pi}} \left\{ 1 + \coth X - \frac{\pi}{\beta} \frac{e^{\beta X/\pi} - 1}{\sinh^2 X} \right\} ,$$

where A is a factor non depending on λ and β . We have to do the limit as $\lambda \rightarrow 0^+$, i.e., $X \rightarrow 0$ in the right hand side of the above equation now. Some trivial calculations lead to:

$$\frac{d}{d\lambda} \lambda^2 W_\beta^+(x_{H^+}, x_{H^+} + \lambda z) = \frac{A}{\beta} \frac{(1 - \beta/2\pi)X^2 + O(X^3)}{X^2 + O(X^3)} \rightarrow A \left(\frac{1}{\beta} - \frac{1}{2\pi} \right) .$$

If we impose that the right hand side of the above equation vanishes as $\lambda \rightarrow 0^+$, i.e., $X \rightarrow 0$, we will recover a Hessling result [8]. Considering the points on the horizon which do not belong to their intersection, the *Hessling principle* holds *only if* the temperature is the *Unruh temperature* ($T = 1/\beta = 2\pi$)⁸.

Finally, we stress that if one considers more complicated geodesics, i.e., geodesics with $x^\perp \neq \text{constant}$, all the above results will not change, but the necessary calculations are more complicated and we will not report on this here.

2 The Case of an Extremal Reissner-Nordström Black Hole

Let us apply the method of the first part on the case of a massless scalar field in the 4-dimensional Reissner-Nordström background. We start with coordinate frames (and related approximations) used in [22].

The metric we are interested in reads:

$$ds^2 = - \left(1 - \frac{R_H}{R} \right)^2 (dx^0)^2 + \left(1 - \frac{R_H}{R} \right)^{-2} dR^2 + R^2 d\Omega_2 , \quad (40)$$

where we are using polar coordinates, R being the radial one and $d\Omega_2$ being the metric of the unit-2-sphere. The horizon radius is $R_H = MG = Q$, M being the mass of the black hole, G the Newton constant and Q its charge.

⁸One should observe that also the limit case $\beta \rightarrow +\infty$ does not satisfy the Hessling condition.

It is obvious that any redefinition of space-like coordinates does not change the thermal properties of our field theory because these properties depend on the time-like coordinate and in particular on its tangent Killing vector. Thus, as in [22], we redefine the Reissner-Nordström radial coordinate by means of

$$\rho = \frac{(\bar{R} - 1)}{1 - 2(\bar{R} - 1) \ln(\bar{R} - 1) - (\bar{R} - 1)^2}, \quad (41)$$

where $\bar{R} = R/R_H$ is now implicitly defined by Eq. (41). Near the horizon $\rho \sim 0$ we can expand:

$$\bar{R} \sim 1 + \rho + O(\rho^2 \ln \rho). \quad (42)$$

In order to perform explicit computations, we shall consider the large mass limit of the black hole, i.e., $R_H \rightarrow +\infty$. In this limit, the whole region outside of the black hole ($R > R_H$) tends to approach the horizon. Thus we can use the approximated metric near the horizon:

$$ds^2 \sim -\rho^2(dx^0)^2 + \frac{1}{\rho^2}d\rho^2 + d\Omega_2. \quad (43)$$

We will return on this point in the final discussion.

Finally we change the space-like frame by a new space-like coordinate:

$$r = \frac{1}{\rho}. \quad (44)$$

The final form of the metric is very simple, it is called the *Bertotti-Robinson metric* [23]:

$$ds^2 = \frac{1}{r^2} \left[-dt^2 + dr^2 + r^2 d\Omega_2 \right], \quad (45)$$

or

$$ds^2 = \frac{1}{\mathbf{x}^2} \left[-dt^2 + (d\mathbf{x})^2 \right], \quad (46)$$

where we used the obvious notation

$$t = x^0,$$

$$\mathbf{x} \equiv (x^1, x^2, x^3).$$

Note that this metric is *conformal* to Minkowski metric by the factor $1/r^2$, *singular* at the origin; however we are interested in the region near the horizon, i.e. $r \rightarrow +\infty$, and this singularity is absent there.

At the beginning our method involves the calculation of all the possible geodesics which start from the horizons. It can be easily proved that this means that we have to look for space-like and time-like geodesics which, starting from the outside of the black hole reach the region $r = \infty$ in a *finite* interval of geodesic length. In particular we seek the geodesics which reach at least one of the limit regions $r = \infty$ and $t = +\infty$ (*future* horizon) or $r = \infty$ and $t = -\infty$ (*past* horizon) in a *finite* interval of geodesic length.

Really, we will find that there exists also a kind of space-like geodesics with $t = \text{constant}$ which just “seem to fall into the horizons”, i.e., they employ an *infinite* geodesic length to

reach the region at $r = \infty$; obviously this kind of geodesics could reach the intersection of the past and the future horizons at most, because of the time coordinate which remains constant. Exactly, the strange behaviour of the above geodesics arises from the fact that the intersection of the two horizons is not contained in the whole manifold, in other words *the future and the past horizons do not intersect at all*, and thus these strange geodesics get lost into infinity.

This is a well known feature of the extremal Reissner-Nordström space-time [24, 26, 27].

The geodesic equations of the metric (45) are:

$$\frac{d}{ds} \left(\frac{1}{r^2} \frac{dt}{ds} \right) = 0 \quad (47)$$

$$\frac{d}{ds} \left(\frac{1}{r^2} \frac{dr}{ds} \right) = -\frac{1}{r^3} \left[\left(\frac{dr}{ds} \right)^2 - \left(\frac{dt}{ds} \right)^2 \right] \quad (48)$$

$$\frac{d}{ds} \left(\frac{d\theta}{ds} \right) = \sin \theta \cos \theta \left(\frac{d\phi}{ds} \right)^2 \quad (49)$$

$$\frac{d}{ds} \left(\sin^2 \theta \frac{d\phi}{ds} \right) = 0 \quad (50)$$

By an opportune choice of the affine parameter, posing s to be equal to the Riemannian length, we may add another equation to the previous set of equations:

$$\frac{1}{r^2} \left[\left(\frac{dr}{ds} \right)^2 - \left(\frac{dt}{ds} \right)^2 \right] + \left(\frac{d\theta}{ds} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{ds} \right)^2 = \pm 1, \quad (51)$$

where the right hand member is equal to $+1$ for space-like geodesics or -1 for time-like geodesics.

Note that, because of the spherical symmetry, every geodesic defines by its space-like components r, θ, ϕ a spatial curve which lies in a space-like 3-plane which intersects the (spatial) origin. Indeed we can consider the geodesic variational equations as the *Euler-Lagrange* equations of motion of a material point, by using the time t as the evolution parameter. Because of the spherical symmetry of the action, using *Noether's* theorem, one concludes that there exists a space-like 3-vector, perpendicular to the spatial motion, which remains constant in time. In fact from Noether's theorem it arises:

$$\varepsilon_{ijk} \frac{\partial L}{\partial \dot{x}^j} \dot{x}^k = \text{constant}, \quad (52)$$

where we used the dot to indicate the time derivative. Obviously we supposed also:

$$L = \sqrt{|g_{00} + g_{pq} \dot{x}^p \dot{x}^q|}, \quad (53)$$

where:

$$g_{00} = \frac{1}{r^2} \quad \text{and} \quad g_{pq} = \frac{\delta_{pq}}{r^2}.$$

If we take the time derivative of (52) and remember the Lagrangean form of the equation of motion we will obtain:

$$\varepsilon_{ijk} \frac{\partial L}{\partial x^j} \dot{x}^k + \varepsilon_{ijk} \frac{\partial L}{\partial \dot{x}^j} \ddot{x}^k = 0.$$

It can easily be shown, using the antisymmetry of ε_{ijk} and the explicit form of the \dot{x}^j -derivative, that the second term of the left hand member vanishes. It remains:

$$\varepsilon_{ijk} \frac{\partial L}{\partial x^j} x^k = 0 \quad \text{i.e.} \quad \frac{\partial L}{\partial \mathbf{x}} \wedge \mathbf{x} = 0.$$

As we argued this is the equation of a plane which intersects the origin.

Once again, because of the spherical symmetry of the problem, we may consider only those geodesics which lie in the plane $\theta = \pi/2$.

In this way the solutions of Eq. (49) and Eq. (50) are trivial:

$$\begin{aligned} \theta &= \frac{\pi}{2}, \\ \phi &= Bs + \phi_0, \end{aligned} \tag{54}$$

where B and ϕ_0 are real constants. Consequently, we can write down Eq. (51) as:

$$\frac{1}{r^2} \left[\left(\frac{dr}{ds} \right)^2 - \left(\frac{dt}{ds} \right)^2 \right] + B^2 = \pm 1. \tag{55}$$

After some trivial calculations and the use of Eq. (55), we end up with the following four sets of solutions of the above differential equations, for all the possible geodesics of the indicated plane which intersect (or seem to do so) the horizons.

Time constant geodesics which seem to reach the horizons.

$$r = r_0 e^{\pm s}, \tag{56}$$

$$t = t_0, \tag{57}$$

$$\phi = Bs + \phi_0, \tag{58}$$

where $r_0 > 0$, t_0 , B , ϕ_0 are real numbers and s is the geodesic length measured from the point $(t_0, r_0, \theta = \pi/2, \phi_0)$. Note that all these geodesics are space-like. We stress that these geodesics *seem* to reach the horizons as $s \rightarrow \pm\infty$, but that is an obvious contradiction; actually, as discussed above, this is a consequence of the fact that $H^+ \cap H^- = \emptyset$.

First family of geodesics which start from the horizons.

$$r = \frac{A}{s}, \tag{59}$$

$$t = t_0 \pm \frac{A}{s}, \tag{60}$$

$$\phi = s + \phi_0, \tag{61}$$

where $A > 0$, t_0 , ϕ_0 are real numbers. The origin of the parameter s is chosen in a way that the starting point of the geodesic is on the horizon. This horizon will be H^+ if the sign in front of A in Eq. (60) is $+$, otherwise it will be H^- if the sign is $-$.

All these geodesics are *space-like*.

Second family of geodesics which start from the horizons.

$$r = \frac{A}{\sin(\sqrt{K}s)}, \quad (62)$$

$$t = t_0 \pm A \cot(\sqrt{K}s), \quad (63)$$

$$\phi = Bs + \phi_0, \quad (64)$$

where $K > 0$, $A > 0$, B , t_0 , ϕ_0 are real numbers; furthermore K and B are related in only one of the following two possibilities:

$K = 1 + B^2$ and the geodesic is *time-like*,

or

$B^2 > 1$, $K = B^2 - 1$ and the geodesic is *space-like*.

The origin of the parameter s is chosen in a way that the starting point of the geodesic is on the horizon. This horizon will be H^+ if the sign in front of A in Eq. (63) is $+$, otherwise it will be H^- if the sign is $-$.

Third family of geodesics which start from the horizon.

$$r = \frac{A}{\sinh(\sqrt{K}s)}, \quad (65)$$

$$t = t_0 \pm A \coth(\sqrt{K}s), \quad (66)$$

$$\phi = Bs + \phi_0, \quad (67)$$

where $K > 0$, $A > 0$, B , t_0 , ϕ_0 are real numbers; furthermore $B^2 < 1$, $K = 1 - B^2$ and all geodesics are space-like.

As in the previous case the origin of the parameter s is chosen in a way that the starting point of the geodesic is on the horizon. This horizon will be H^+ if the sign in front of A in Eq. (66) is $+$, otherwise it will be H^- if the sign is $-$.

Now we calculate the Wightman function for a massless scalar field in the metric (45). This metric is conformal to the Minkowski metric, thus we may use the Dowker and Schofield's method [13, 25] which connects the Wightman functions (in general the Green functions) of a scalar field in a static manifold with the corresponding Wightman functions of the field in another, *conformally* related static manifold.

Let us suppose to have two *static* metrics which are conformally related:

$$ds^2 = g_{00}(\mathbf{x})(dx^0)^2 + g_{ij}(\mathbf{x})dx^i dx^j \quad (68)$$

and

$$ds'^2 = g'_{00}(\mathbf{x})(dx^0)^2 + g'_{ij}(\mathbf{x})dx^i dx^j, \quad (69)$$

where

$$g'_{\mu\nu} = \lambda^2(\mathbf{x})g_{\mu\nu} \quad (70)$$

and let us consider the Wightman functions which satisfy the respective Klein-Gordon equations:

$$(\square + \xi R + m^2)W_{\beta}^{\pm}(x, x') = 0 \quad (71)$$

and

$$\left(\square' + \xi R' + \left(\xi - \frac{1}{6} \right) \square' (\lambda^{-2}) + m^2 \lambda^{-2} \right) W_{\beta}^{\pm}(x, x') = 0, \quad (72)$$

where R is the scalar curvature.

The Wightman functions above are related by the Dowker-Schofield scaling property:

$$W_{\beta}^{\pm}(x, x') = \lambda^{-1}(\mathbf{x}) W_{\beta}^{\pm}(x, x') \lambda^{-1}(\mathbf{x}'), \quad (73)$$

In our case ds'^2 is the Minkowski metric (hence $R' = 0$), ds^2 is the metric of Eq. (46) and thus $\lambda^2 = \mathbf{x}^2$.

The Wightman functions which we are interested in satisfy the Klein-Gordon equation in (71) with $m = 0$ and $R = 0$, in fact the metric of Eq. (46) (but also the real extremal R-N metric) has vanishing scalar curvature. We stress that, due to above fact, the *minimal* coupling in Eq.(71) coincides with the *conformal* coupling. Thus we can choose the value of the parameter ξ to be $\xi = 1/6$ (conformal coupling) in Eq.(71) and hence in Eq.(72) which reduces to the usual massless K-G equation in the Minkowski space-time.

The thermal Wightman functions of a massless scalar field in Minkowski space can be obtained by using the procedure sketched in **Section 1**. They read:

$$W_{M\beta}^{\pm} = \frac{1}{4\pi^2} \frac{\pi \left\{ \coth \frac{\pi}{\beta} (|\mathbf{x} - \mathbf{x}'| + t - t' \mp i\varepsilon) + \coth \frac{\pi}{\beta} (|\mathbf{x} - \mathbf{x}'| - t + t' \pm i\varepsilon) \right\}}{2\beta |\mathbf{x} - \mathbf{x}'|} \quad (74)$$

Using Eq.(73) we find the thermal Wightman functions of a massless scalar field (minimally as well as conformally coupled) propagating outside of a large mass, extremal Reissner-Nordström black-hole:

$$W_{\beta}^{\pm} = \frac{|\mathbf{x}| |\mathbf{x}'|}{4\pi^2} \frac{\pi \left\{ \coth \frac{\pi}{\beta} (|\mathbf{x} - \mathbf{x}'| + t - t' \mp i\varepsilon) + \coth \frac{\pi}{\beta} (|\mathbf{x} - \mathbf{x}'| - t + t' \pm i\varepsilon) \right\}}{2\beta |\mathbf{x} - \mathbf{x}'|} \quad (75)$$

If we take the limit $|\mathbf{x}| \rightarrow +\infty$ we obtain the thermal Wightman functions calculated on the horizon (in the argument x). In order to calculate this limit we must increase (to reach H^+) or decrease (to reach H^-) the variable t together the variable \mathbf{x} and ϕ along the geodesics obtained above. Once again we deal with H^+ only because of the time symmetry of the problem.

If we consider either the space-like or the time-like geodesics of the above three families we will produce the same function:

$$W_{\beta}^{\pm}(x_{H^+}, x') = \frac{r' \pi \left\{ 1 + \coth \left[\frac{\pi}{\beta} (t' - t_0 - r' \cos(\phi' - \phi_0)) \right] \right\}}{4\pi^2 2\beta} \quad (76)$$

Note that along the curves satisfying:

$$r' \cos(\phi' - \phi_0) = t' - t_0,$$

there is a divergence. Indeed it can be easily proved, starting from the metric of Eq. (45), that all the *light-like* geodesics which meet horizon H^+ satisfy this relation, thus this is the correct, expected divergence.

In order to check the HNS principle on H^+ for a space-like vector z tangent to a geodesic of the first family in the horizon, we consider Eq. (76) and substitute the variables

r', t', ϕ' for the functions defined in the right hand side of equations (59), (60), (61), using also the identifications: $t_0 = t'_0$ and $\phi_0 = \phi'_0$. Finally we redefine $s = \lambda z^i$.

We obtain, as $\lambda \rightarrow 0$:

$$\begin{aligned}
\lambda^2 W_\beta (x_{H^+}, x_{H^+} + \lambda z) &\sim \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \lambda z^i} \left\{ \coth \left[\frac{\pi}{\beta} \left(A \frac{1}{\lambda z^i} - \frac{A \cos(\lambda z^i)}{\lambda z^i} \right) \right] + 1 \right\} \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \lambda z^i} \left\{ \sinh \left[\frac{\pi}{\beta} \left(A \frac{1}{\lambda z^i} - \frac{A \cos(\lambda z^i)}{\lambda z^i} \right) \right] \right\}^{-1} \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \lambda z^i} \left\{ \frac{\pi}{\beta} \left(A \frac{1}{\lambda z^i} - \frac{A \cos(\lambda z^i)}{\lambda z^i} \right) \right\}^{-1} \\
&\sim \frac{\lambda^2}{4\pi^2 2 z^i} \frac{z^i}{\frac{1}{2}(\lambda z^i)^2} = \frac{1}{4\pi^2} \frac{1}{g_{\mu\nu}(H^+)} \frac{1}{z^\mu z^\nu}.
\end{aligned}$$

We conclude that in this case the HNS principle holds for any value of β .

In order to check the HNS principle on H^+ for a time-like vector z tangent to a geodesic of the second family on the horizon, we consider Eq. (76) and substitute the variables r', t', ϕ' for the functions defined in the right hand side of equations (62), (63), (64), with the identifications: $t_0 = t'_0$ and $\phi_0 = \phi'_0$. Finally we redefine $s = \lambda z^0$.

We obtain, as $\lambda \rightarrow 0$:

$$\begin{aligned}
\lambda^2 W_\beta (x_{H^+}, x_{H^+} + \lambda z) &\sim \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \sin(\lambda\sqrt{K}z^0)} \left\{ \coth \left[\frac{\pi}{\beta} \left(A \cot(\lambda\sqrt{K}z^0) - \frac{A \cos(B\lambda z^0)}{\sin(\lambda\sqrt{K}z^0)} \right) \right] + 1 \right\} \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \sin(\lambda\sqrt{K}z^0)} \left\{ \sinh \left[\frac{\pi}{\beta} \left(A \cot(\lambda\sqrt{K}z^0) - \frac{A \cos(B\lambda z^0)}{\sin(\lambda\sqrt{K}z^0)} \right) \right] \right\}^{-1} \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \sin(\lambda\sqrt{K}z^0)} \left\{ \frac{\pi}{\beta} \left(A \cot(\lambda\sqrt{K}z^0) - \frac{A \cos(B\lambda z^0)}{\sin(\lambda\sqrt{K}z^0)} \right) \right\}^{-1} \\
&\sim \frac{1}{4\pi^2 2\sqrt{K}z^0} \left\{ \frac{\cos(\lambda\sqrt{K}z^0) - \cos(B\lambda z^0)}{\lambda \sin(\lambda\sqrt{K}z^0)} \right\}^{-1} \\
&\sim \frac{\lambda}{4\pi^2 \sqrt{K}} \frac{\lambda\sqrt{K}}{(\lambda B z^0)^2 - (\lambda\sqrt{K}z^0)^2} \\
&\sim \frac{1}{4\pi^2} \frac{1}{z^{02} (B^2 - K)} = \frac{1}{4\pi^2} \frac{1}{g_{\mu\nu}(H^+)} \frac{1}{z^\mu z^\nu}
\end{aligned}$$

In the last step we used $B^2 - K = -1$. We conclude that in this case the HNS principle holds for any value of β .

For space-like vectors the calculations are quite identical except for the fact that one has to substitute z^0 for z^i above and that it holds $B^2 - K = +1$ instead of $B^2 - K = -1$.

In order to check the HNS principle on H^+ for a space-like vector z tangent to a geodesic of the third family on the horizon, we consider Eq. (76) and we substitute the variables r', t', ϕ' for the functions defined in the right hand of equations (65), (66), (67), with the identifications: $t_0 = t'_0$ and $\phi_0 = \phi'_0$. Finally we redefine $s = \lambda z^i$. We obtain, as $\lambda \rightarrow 0$:

$$\begin{aligned}
\lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) &\sim \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \sinh(\lambda\sqrt{K}z^i)} \left\{ \coth \left[\frac{\pi}{\beta} \left(A \coth(\lambda\sqrt{K}z^i) - \frac{A \cos(B\lambda z^i)}{\sinh(\lambda\sqrt{K}z^i)} \right) \right] + 1 \right\} \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \sinh(\lambda\sqrt{K}z^i)} \left\{ \sinh \left[\frac{\pi}{\beta} \left(A \coth(\lambda\sqrt{K}z^i) - \frac{A \cos(B\lambda z^i)}{\sinh(\lambda\sqrt{K}z^i)} \right) \right] \right\}^{-1} \\
&\sim \frac{A\pi\lambda^2}{4\pi^2 2\beta \sinh(\lambda\sqrt{K}z^i)} \left\{ \frac{\pi}{\beta} \left(A \coth(\lambda\sqrt{K}z^i) - \frac{A \cos(B\lambda z^i)}{\sinh(\lambda\sqrt{K}z^i)} \right) \right\}^{-1} \\
&\sim \frac{1}{4\pi^2 2\sqrt{K}z^i} \left\{ \frac{\cosh(\lambda\sqrt{K}z^i) - \cos(B\lambda z^i)}{\lambda \sinh(\lambda\sqrt{K}z^i)} \right\}^{-1} \\
&\sim \frac{\lambda}{4\pi^2 \sqrt{K}} \frac{\lambda\sqrt{K}}{(\lambda B z^i)^2 + (\lambda\sqrt{K}z^i)^2} \\
&\sim \frac{1}{4\pi^2} \frac{1}{z^{i2} (B^2 + K)} = \frac{1}{4\pi^2} \frac{1}{g_{\mu\nu}(H^+) z^\mu z^\nu}
\end{aligned}$$

In the last step we used $B^2 + K = +1$. We conclude that in this case the HNS principle holds for any value of β .

We finally stress that the HNS principle accepts the limit value of the temperature $T = 0$, too. Indeed, starting from Wightman functions in Minkowski space-time we find the *zero temperature* Wightman functions in our coordinate by using Eq. (73). They read as:

$$W^\pm(x, x') = \frac{|\mathbf{x}| |\mathbf{x}'|}{4\pi^2} \frac{1}{\sigma^2(x_{\pm\epsilon}, x')}.$$

The same result arises doing the limit as $\beta \rightarrow +\infty$ in Eq.(75).

By putting an argument on the horizon we obtain:

$$W^\pm(x_{H^+}, x') = \frac{1}{8\pi^2} \left(\frac{t' - t_0}{r'} - \cos(\phi' - \phi_0) \right)^{-1}. \quad (77)$$

Once again we may observe that one obtains the same result doing the limit as $\beta \rightarrow +\infty$ in Eq.(76) directly.

Now it is really very easy to check, following the usual procedure, that the above Wightman functions satisfy the HNS principle on the horizons for all time-like or space-like geodesics which reach the horizons and they diverge for light-like ones.

Finally, let us check the *Hessling principle* in the present case.

We consider the space-like geodesics of the first family in Eq.s (59), (60) and (61). It arises by using that family of geodesics and Eq.(76):

$$\frac{d}{d\lambda} \lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) = \frac{\Gamma}{\beta} \frac{d}{d\lambda} \lambda \left\{ 1 + \coth \left[\frac{\pi A}{\beta \lambda z} (1 - \cos \lambda z) \right] \right\},$$

where the factor $\Gamma := A/8\pi z$ is *finite* and it does not depend on λ and β . Some trivial calculations lead to:

$$\frac{d}{d\lambda} \lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) = \frac{\Gamma}{\beta} \left\{ 1 + \frac{X + \sinh X \cosh X - \frac{\pi A}{\beta z} \sin \lambda z}{\sinh^2 X} \right\},$$

where we also posed:

$$X := \frac{\pi A}{\beta z} \frac{1 - \cos \lambda z}{\lambda}.$$

Expanding around $\lambda = 0$, i.e., $X = 0$ it arises:

$$\frac{d}{d\lambda} \lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) = \frac{\Gamma}{\beta} \left\{ 1 + \frac{O(\lambda^3)}{\lambda^2 + O(\lambda^4)} \right\},$$

Hence we get, doing the limit as $\lambda \rightarrow 0^+$:

$$\frac{d}{d\lambda} \lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) \rightarrow \frac{\Gamma}{\beta}.$$

This fact is sufficient to prove that the Hessling principle excludes every finite value of β . The limit case $T = 1/\beta = 0$ survives only. Using the remaining two families of geodesics, calculations result to be very similar and the same limit value of the temperature survives. Furthermore, it can be simply proved that the Wightman functions of the R-N vacuum, i.e., the limit case $T = 1/\beta = 0$, satisfies the Hessling principle by considering directly Eq.(77). In fact it follows from Eq.(77) using geodesics of the first family:

$$\begin{aligned} \frac{d}{d\lambda} \lambda^2 W(x_{H^+}, x_{H^+} + \lambda z) &= \\ &= \frac{1}{8\pi^2} \frac{2\lambda - 2\lambda \cos(\lambda z^i) - z^i \lambda^2 \sin(\lambda z^i)}{(1 - (\lambda z^i))^2} = \frac{1}{8\pi^2} \frac{O(\lambda^5)}{O(\lambda^4)} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0^+. \end{aligned}$$

By using the geodesic of the second family we obtain similarly:

$$\begin{aligned} \frac{d}{d\lambda} \lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) &= \\ &= \frac{1}{8\pi^2} \frac{2\lambda}{\cos(\lambda z \sqrt{K}) - \cos(\lambda z B)} + \frac{\lambda^2 (\sqrt{K} z \sin(\lambda z \sqrt{K}) - B z \sin(\lambda z B))}{(\cos(\lambda z \sqrt{K}) - \cos(\lambda z B))^2} = \\ &= \frac{\lambda O(\lambda^4)}{O(\lambda^4)} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0^+. \end{aligned} \tag{78}$$

Finally, using the third family of geodesics:

$$\begin{aligned} \frac{d}{d\lambda} \lambda^2 W_\beta(x_{H^+}, x_{H^+} + \lambda z) &= \\ &= \frac{1}{8\pi^2} \frac{2\lambda}{\cosh(\lambda z \sqrt{K}) - \cos(\lambda z B)} - \frac{\lambda^2 (\sqrt{K} z \sinh(\lambda z \sqrt{K}) + B z \sin(\lambda z B))}{(\cosh(\lambda z \sqrt{K}) - \cos(\lambda z B))^2} = \\ &= \frac{\lambda O(\lambda^4)}{O(\lambda^4)} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0^+. \end{aligned} \tag{79}$$

3 Discussion

The most important conclusion which follows from the above calculations is that the *weak* HNS principle, in the case of an extreme Reissner-Nordström black hole, holds for every value of β , i.e., once again it agrees with the other method based on the elimination of the singularities of the Euclidean manifold, but the *weak* Hessling principle selects only the null temperature, i.e., the R-N vacuum as a physically sensible state.

We observe that our “along geodesics” calculations eliminate quantum states which do not have the correct scaling limit, on the other hand one can not correctly think that they determine only the states which have the correct scaling limit in the sense precised in [8]. In fact we used a *weaker* prescription as previously stressed.

Another important point is that we dealt with the limit of a large mass black hole and with a massless field, but we think that our conclusions should hold without to assume these strong conditions, too.

Indeed, in order to check the behaviour of Wightman functions on the horizon we recognize that is sufficient to know the form of the Wightman functions *near the horizon* only. In this region, regardless of the value of black hole’s mass, the metric can be written in the form used above:

$$ds^2 \sim \frac{1}{r^2} \left[-dt^2 + dr^2 + r^2 d\Omega_2 \right] ,$$

thus we expect that the Wightman functions for arguments near the horizon should be of the form (75) and thus it should be possible to restore all our results. For example, we stress that Haag, Narnhofer and Stein in [6] used just the limit form of the metric near the horizon in order to obtain the Hawking temperature. However, one could object that the normalization of the modes used to construct the Wightman functions depends on the integration over the whole spatial manifold and not only on the region near the horizon. Really, it is possible to overcome this problem at least formally dealing with our static metric. In fact, in this case one recovers by the KMS condition [6] ($x \equiv (\tau, \mathbf{x})$):

$$\langle \phi(x_1) \phi(x_2) \rangle_\beta = \frac{i}{2\pi} \int_{-\infty}^{+\infty} G(\tau_1 + \tau, \mathbf{x}_1 \mid \tau_2, \mathbf{x}_2) \frac{e^{\beta\omega}}{e^{\beta\omega} - 1} e^{i\omega\tau} d\tau d\omega , \quad (80)$$

where the distribution G is the *commutator* of the fields and thus it is uniquely determined [6] by the fact that it is a solution of the Klein-Gordon equation in both arguments, vanishes for equal times $\tau_1 = \tau_2$ and is normalized by the “local” condition:

$$g^{\tau\tau} \sqrt{-g} \frac{\partial}{\partial \tau_1} G(x_1, x_2) \mid_{\tau_1=\tau_2} = \delta^3(\mathbf{x}_1, \mathbf{x}_2) \quad (81)$$

The above 3-delta function is usually understood as:

$$\begin{aligned} \delta^3(\mathbf{x}_1, \mathbf{x}_2) &= 0 \quad \text{for} \quad \mathbf{x}_1 \neq \mathbf{x}_2 \\ \int \delta^3(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_2 &= 1 \end{aligned}$$

By the previous, spatially “local” formulas we expect that the function G calculated by using the “true” static metric becomes the function G calculated by using the approximated static form of the metric inside of a certain static region $\delta\Sigma \times \mathbb{R}$ (where $\tau \in \mathbb{R}$) as $\delta\Sigma$ shrinks around a 3-point. Really, considering $(\delta\Sigma, \tau_0)$ as a *Cauchy surface*, the above

result should come out inside of the “diamond-shaped” 4–region causally determined by $(\delta\Sigma, \tau_0)$ at least. But, studying the form of the light cones near event horizons of the form $|\mathbf{x}| = r_0, \tau \in \mathbb{R}$ (including the limit case $r_0 \rightarrow \pm\infty$), it is simple to prove that this 4–region will tend to contain the whole τ –axis if $\delta\Sigma$ approaches to the event horizons.

This is the case of an extremal R-N black hole where the Bertotti-Robinson metric approximates the R-N metric near the horizon $r \rightarrow +\infty, t \in \mathbb{R}$.

In the same way, using Eq. (80), we could expect such a property for thermal Wightman functions, too, the case of zero temperature, which is regarded as the limit $\beta \rightarrow +\infty$, included. Furthermore, if the field’s mass m in Eq. (71) were not zero (and we chose again $\xi = 1/6$ following the previous motivations), the Wightman functions which we are interested in would be connected to the Wightman functions satisfying the ordinary Klein-Gordon equation in the Minkowski space-time except for a position dependent mass term (see Eq. (72)):

$$M^2(r) = \frac{m^2}{r^2}.$$

We stress that this mass term is vanishing as $r \rightarrow +\infty$. Our method works for large value of r so we expect that our conclusions do not change in the case of a *massive* scalar field. Finally, we stress that recently P.H. Anderson, W.A. Hiscock and D.J Loran [28], by using of the metric in Eq.(45) and the Brown-Cassidy-Bunch formula (see [28, 10] and references therein) argued (and numerically checked by using the *complete* R-N metric) that the Reissner-Nordström vacuum state is the *only* thermal state with a non-singular renormalized stress-tensor on the horizon of an extremal R-N black hole. In fact they obtained the formula holding near the horizon:

$$\langle T_\mu^\nu \rangle_{\beta renorm.} \sim \frac{1}{2880\pi^2} \delta_\nu^\mu + r^4 \frac{\pi^2}{30\beta^4} \text{diag} \left(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \quad (82)$$

Note that, if $T = 1/\beta$ does not vanish, there will be a strong divergence as $r \rightarrow +\infty$, i.e., on the horizon. This divergence is not due to the coordinate frame used because it remains also for scalar quantities as $T_\mu^\nu T_\nu^\mu$. Supposing the stress-tensor generates the gravity by means of Einstein’s equations (or by some similar generalisation), the above divergence generates a singularity in the metric structure of the manifold. In the framework of the *Semiclassical Quantum Gravity* (see [10] for example) the R-N vacuum state results to be the only possible state in equilibrium with an extremal R-N black hole.

We observe that the “improved” HNS prescription, i.e. the Hessling principle agrees completely with the result of Anderson Hiscock and Loran in our weaker formulation at least, in particular it selects a state carrying a renormalized stress-tensor finite on the horizon. This fact comes out also both in the Rindler space where the HNS and Hessling’s prescriptions selects the Minkowski vacuum which has a regular stress tensor on the horizon or in the Schwarzschild space where the HNS principle selects the Hartle-Hawking state with the same property on the horizon [10].

Acknowledgements

I would like to thank G.Cognola, M.Toller, L.Vanzo and S.Zerbini of the Dipartimento di Fisica dell’Università di Trento for all the useful discussions about the topics of this

paper.

I am grateful to Bernard Kay and a referee of *Classical and Quantum Gravity* who pointed out the Hessling paper to me.

Finally, I would like to thank S. Steidl of the Physics Department of the University of Innsbruck for the constant and patient technical assistance.

References

- [1] Hawking S W, Horowitz G T, Ross S F 1995 Entropy, Area, and Black Hole Pairs *Phys.Rev.* **D 51** 4302, *gr-qc/9409013*
- [2] Narlikar J V and Padmanabhan T 1986 *Gravity, Gauge Theories and Quantum Cosmology* (Dordrecht: D.Reidel Publishing Company)
- [3] Kubo R 1957 *J. Math. Soc. Jpn.* **12** 570,

Martin P C Schiwinger J 1959 *Phys. Rev.* **115** 1342,

Haag R, Hugenholtz N M and Winnink M 1967 *Commun. Math. Phys* **5** 215
- [4] Haag R 1992 *Local Quantum Physics* (Berlin: Springer-Verlag)
- [5] Bertola M, Gorini V, Zeni M 1995 Quantized Temperature Spectra in Curved Space-time *preprint Universita' di Milano* DYSCO 056,IFUM 513/FT *hep-th/9508004*
- [6] Haag R, Narnhofer H, Stein U 1984 *Commun. Math. Phys.* **94** 219
- [7] Fredenhagen K Haag R 1987 *Commun. Math. Phys.* **108** 91
- [8] Hessling H 1994 *Nuc. Phys.* **B 415** 243
- [9] Kay B S and Wald R M 1991 *Phys. Rep.* **207** 49
- [10] Birrel N D and Davies P C W 1982 *Quantum Field Theory in Curved Space.* (Cambridge: Cambridge University Press)
- [11] Fulling S A 1989 *Aspects of Quantum Field Theory in Curved Space-Time.* (Cambridge: Cambridge University Press)
- [12] S.A. Fulling and S.N.M. Ruijsenaars. *Phys. Rep.* **152**, 135 (1987).
- [13] Kennedy G, Critchley R C, Dowker J S 1980 *Ann. Phys.* **125** 346
- [14] Dowker J S 1977 *J. Phys.* **A10** 115
- [15] Dowker J S 1978 *Phys. Rev.* **18** 1856
- [16] Dowker J S 1987 *Phys. Rev. D* **36** 3095
- [17] Hawking S W 1978 *Phys. Rev.***D16** 1747
- [18] Dowker J S 1987 *Phys. Rev. D* **36** 3742
- [19] Cognola G, Kirsten K, Vanzo L 1994 *Phys. Rev. D* **49** 1029
- [20] Solodukhin G 1995 The Conical Singularity and Quantum Corrections to BH Entropy, *Phys.Rev.***D 51** 609, *hep-th/9407001*
- [21] Takagi S 1986 *Progr. Theor. Phys. Suppl.* **88**

- [22] Cognola G, Vanzo L, Zerbini S 1995 One-loop Quantum Corrections to the Entropy for an Extremal Reissner-Nordström Black-Hole *Phys.Rev.D* **52** 4548, *hep-th/9504064*
- [23] Bertotti B 1959 *Phys. Rev.* **116** 1331,
 Robinson I 1959 *Bull. Acad. Pol. Sci.* **7** 351
- [24] Carter B 1996 *Phys. Rev. Lett.* **21** 423
- [25] Dowker J S and Schofield J P 1988 *Phys. Rev. D* **38** 3327
- [26] Wald R N 1984 *General Relativity* (Chicago: The University of Chicago Press)
- [27] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of the Space-Time.* (Cambridge: Cambridge University Press)
- [28] Anderson P R, Hiscock W A, Loranz D J 1995 *Phys. Rev. Lett.* **74** 4365, *gr-qc/9504019*